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Exact spin dynamics of the $1/r^2$ supersymmetric *t–J* model in a magnetic field

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Abstract
The dynamical spin structure factor $S^{zz}(Q,\omega)$ in the small momentum region is derived analytically for the one-dimensional supersymmetric *t–J* model with $1/r^2$ interaction. Strong spin–charge separation is found in the spin dynamics. The structure factor $S^{zz}(Q,\omega)$ with a given spin polarization does not depend on electron density in the small momentum region. In the thermodynamic limit, only two spinons and one antispinon (magnon) contribute to $S^{zz}(Q,\omega)$. These results are derived via solution of the *SU*(2,1) Sutherland model in the strong coupling limit.

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1. Introduction

Spin–charge separation is a key subject in one-dimensional interacting electron systems. Conformal field theory has succeeded in the description of spin–charge separation in the low-energy physics of the Tomonaga–Luttinger liquid. Beyond the conformal field theory limit, exactly solvable models provide us with opportunities to obtain analytical knowledge on thermodynamics and dynamics, and it is intriguing how the spin–charge separation appears in these properties.

Among exactly solvable models, the supersymmetric *t–J* model with $1/r^2$ interaction [1] reveals the spin–charge separation in the simplest manner. The Hamiltonian of this model is given by

$$\mathcal{H}_{tJ} = \sum_{i<j}\left[-t_{ij}\sum_{\sigma=\uparrow,\downarrow}\left(\tilde{c}^{\dagger}_{i\sigma}\tilde{c}_{j\sigma}+\text{h.c.}\right)+J_{ij}\left(\boldsymbol{S}_i\cdot\boldsymbol{S}_j-\frac{1}{4}n_in_j\right)\right]-h\sum_j S^z_j, \tag{1}$$

where $\tilde{c}_{i\sigma} = c_{i\sigma}(1-n_{i,-\sigma})$ with $c_{i\sigma}$ being the annihilation operator of an electron with spin σ at site i, and $n_i = \sum_\sigma n_{i,\sigma} = \sum_\sigma c^{\dagger}_{i\sigma}c_{i\sigma}$. The spin operator associated with site i is defined as $\boldsymbol{S}_i = \sum_{\alpha,\beta}c^{\dagger}_{i\alpha}(\boldsymbol{\sigma}_i)_{\alpha\beta}c_{i\beta}/2$ where $\boldsymbol{\sigma} = (\sigma^x,\sigma^y,\sigma^z)$ is the vector of Pauli matrices. The transfer energy t_{ij} and exchange one J_{ij} are given by $t_{ij} = J_{ij}/2 = tD_{ij}^{-2}$ where $D_{ij} = (N/\pi)\sin(\pi(i-j)/N)$ with N being the number of lattice sites. Henceforth we take t as the unit of energy. We note that this t–J model reduces to the Haldane–Shastry spin chain model [2, 3] at half-filling. In the supersymmetric t–J model with $1/r^2$ interaction, exact thermodynamics can be interpreted in terms of free spinons and holons [4, 5]. At low temperature, the spin susceptibility is independent of the electron density $\bar{n}$, and the charge susceptibility is independent of the magnetization $\bar{m}$. These features are referred to as the strong spin–charge separation [4]. In addition, the fact that the magnetization $\bar{m}$ for a certain range of h is independent of $\bar{n}$ can be regarded as the strong spin–charge separation. Namely, $\bar{m}$ is determined by h as follows:

$$\bar{m} = \begin{cases} 1 - \sqrt{1 - 2h/\pi^2}, & \text{for } 0 \leqslant h \leqslant h_c \\ \bar{n}, & \text{for } h \geqslant h_c, \end{cases} \tag{2}$$

where $h_c = \bar{n}(2-\bar{n})\pi^2/2$ [6].

The strong spin–charge separation appears also in dynamics at zero temperature. The dynamical spin structure factor is given by

$$S^{zz}(Q,\omega) = \sum_{\nu} \left|\langle\nu|S_Q^z|0\rangle\right|^2 \delta(\omega - E_\nu + E_0), \tag{3}$$

where $S_Q^z = \sum_l S_l^z \,\mathrm{e}^{\mathrm{i}Ql}/\sqrt{N}$. Here $|\nu\rangle$ denotes a normalized eigenstate of the Hamiltonian with energy E_ν (E_0 being the ground state energy). In the absence of magnetic field ($h = 0$), the dynamical spin structure factor was exactly obtained at $\bar{n} = 1$ [7–9]. It was numerically demonstrated that the weight of the dynamical spin structure factor in the t–J model does not depend on $\bar{n}$ in the region where only two spinons contribute [10]. This is an indication of the strong spin–charge separation in dynamics. Later, a mathematical poof was given to this statement, and the analytical expression of the dynamical spin structure factor for $\bar{n} < 1$ was obtained in the full (Q, ω) space [11].

A numerical study [12] also showed that the strong spin–charge separation for $S^{zz}(Q,\omega)$ can be extended to the case of finite magnetic field ($h \neq 0$). Namely, at fixed magnetization, $S^{zz}(Q,\omega)$ away from half-filling is the same as that for half-filling (i.e., the Haldane–Shastry model), in the region where only spinons and antispinons contribute. For $h \neq 0$, the full exact results on $S^{zz}(Q,\omega)$ have not been obtained even in the Haldane–Shastry model. However, if the momentum is restricted to $Q \leqslant \pi\bar{m}$, the dynamical structure factor $S^{zz}(Q,\omega)$ at $\bar{n} = 1$ can be expressed as the dynamical density–density correlation function of the Sutherland model with coupling parameter $\beta = 2$ [13, 14]. In order to give this expression, we assume the positive magnetization $\bar{m}$ without loss of generality, where $\bar{m} = \bar{n}_\uparrow - \bar{n}_\downarrow$ with $\bar{n}_\sigma = N_\sigma/N$ (N_σ being the number of electrons with σ-spin). In the thermodynamics limit, we have the following expression[3]:

$$S^{zz}(Q,\omega) = \frac{Q^2}{4\pi}\int_0^{\pi(1-\bar{m})} \mathrm{d}q_1 \int_0^{\pi(1-\bar{m})} \mathrm{d}q_2 \int_0^{\pi\bar{m}} \mathrm{d}q_{\mathrm{a}}\, \delta\left(Q - q_{\mathrm{a}} - \sum_{j=1}^{2} q_j\right)$$
$$\times\, \delta\left(\omega - \epsilon_{\mathrm{a}}(q_{\mathrm{a}}) - \sum_{j=1}^{2}\epsilon_{\mathrm{s}}(q_j)\right) \frac{|q_1 - q_2|\epsilon_{\mathrm{a}}(q_{\mathrm{a}})}{\prod_{j=1}^{2}(q_{\mathrm{a}} + 2q_j)^2 \prod_{j=1}^{2}\epsilon_{\mathrm{s}}(q_j)^{1/2}}, \tag{4}$$

where $\epsilon_{\mathrm{s}}(q)$ is the spinon spectrum: $\epsilon_{\mathrm{s}}(q) = q(v_{\mathrm{s}} - q)$, and $\epsilon_{\mathrm{a}}(q)$ is the antispinon spectrum: $\epsilon_{\mathrm{a}}(q) = q(v_{\mathrm{s}} + q/2)$, where $v_{\mathrm{s}} = \pi(1-\bar{m})$. The purpose of this paper is to prove that

[3] For the expression of $S^{zz}(Q,\omega)$ in [14], the authors erroneously typed the integration ranges of the spinon momenta $q_i (i = 1, 2)$ as $0 < q_i < \pi\bar{n}_\downarrow$. They should read $0 < q_i < \pi(1-\bar{m})$, as shown in equation (4).

the analytical expression of $S^{zz}(Q,\omega)$ away from half-filling is the same as equation (4), if $0 < Q \leqslant \min[\pi\bar{m}, \pi\bar{n}_\downarrow]$. This yields a mathematical proof of the strong spin–charge separation in magnetic-field dynamics. We stress that the use of the replica type technique is crucial for calculation of the matrix element in $S^{zz}(Q,\omega)$.

This paper is organized as follows. In the next section, we introduce the $SU(2,1)$ Sutherland model as an auxiliary. As in the previous study of thermodynamics [4, 5, 15, 16] and dynamics [8, 9, 17, 18], we take the limit $\beta \to \infty$ of the coupling parameter in order to obtain the analytical knowledge of the $1/r^2$ supersymmetric t–J model. The eigenfunctions of the Sutherland model can be expressed in terms of Jack polynomials. We discuss the basic features of the Jack polynomials. In section 3, we derive the matrix element of the dynamical spin structure factor based on the replica type technique [19]. In section 4, we present the analytic expression of $S^{zz}(Q,\omega)$ for finite systems. Section 5 is devoted to a summary. In appendix A, we derive the dynamical charge structure factor $N(Q,\omega)$ in the same method. In appendix B, we show the comparison with numerical results for small size systems [12]. In appendix C, we present the results on the static structure factors $S^{zz}(Q)$ and $N(Q)$.

2. Sutherland model with $SU(2,1)$ symmetry

In this section, we introduce the Sutherland model [20–23] with $SU(2,1)$ symmetry [24], and review the basic properties.

2.1. Notation

We follow the notations of [25–27]. For a fixed non-negative integer n, let $\Lambda_n = \{\eta = (\eta_1, \eta_2, \ldots, \eta_n) | \eta_i \in \mathbf{Z}_{\geqslant 0}, 1 \leqslant i \leqslant n\}$ be the set of all compositions with length less than or equal to n. The diagram of a composition $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \Lambda_n$ is defined as the set of points $(i, j) \in \mathbf{Z}^2$ such that $1 \leqslant j \leqslant \eta_i$. The weight $\|\eta\|$ of a composition $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \Lambda_n$ is defined by $\|\eta\| = \sum_{i=1}^n \eta_i$. The length $l(\eta)$ of η is defined as the number of non-zero η_i in η. The set of all partitions with length less than or equal to n is defined by $\Lambda_n^+ = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda_n | \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}$. We also denote a partition λ by $a^{m_a} b^{m_b} c^{m_c} \cdots$ or by $(a^{m_a}, b^{m_b}, c^{m_c}, \ldots)$ with $a > b > c > \cdots \geqslant 0$ where m_i is the number of parts which are equal to i. The conjugate partition λ' of a partition λ is a partition whose diagram is the transposition of the diagram of λ. Hence λ'_i is the number of nodes in the ith column of the diagram of partition λ. In particular we have $\lambda'_1 = l(\lambda)$. We define the subset $\Lambda_n^{+,>}$ of the set Λ_n^+ by $\Lambda_n^{+,>} = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda_n | \lambda_1 > \lambda_2 > \cdots > \lambda_n \geqslant 0\}$. Note that for any element $\lambda \in \Lambda_n^{+,>}$, there exists unique partition $\mu \in \Lambda_n^+$ such that $\lambda = \mu + \tilde{\delta}(n)$ with $\tilde{\delta}(n) = (n-1, n-2, \ldots, 1, 0) \in \Lambda_n^{+,>}$. For two distinct partitions $\lambda, \mu \in \Lambda_n^+$, we define the dominance order $\lambda < \mu$ if $\|\lambda\| = \|\mu\|$ and $\sum_{i=1}^k \lambda_i \leqslant \sum_{i=1}^k \mu_i$ for all $k = 1, \ldots, n$. For a composition $\eta \in \Lambda_n$, η^+ denotes the unique partition which is a rearrangement of the composition η. Now we define a partial order $\prec$ on compositions as follows: for $\nu, \eta \in \Lambda_n$, $\nu \prec \eta$ if $\nu^+ < \eta^+$ with dominance ordering on partitions or if $\nu^+ = \eta^+$ and $\sum_{i=1}^k \nu_i \leqslant \sum_{i=1}^k \eta_i$ for all $k = 1, \ldots, n$.

For a given composition $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ and $s = (i, j)$ in the diagram of the composition η, we define the following quantities:

$$a_\eta(s) = \eta_i - j, \tag{5}$$

$$a'_\eta(s) = j - 1, \tag{6}$$

$$l_\eta(s) = \#\{k \in \{1, \ldots, i-1\} | j \leqslant \eta_k + 1 \leqslant \eta_i\} + \#\{k \in \{i+1, \ldots, n\} | j \leqslant \eta_k \leqslant \eta_i\}, \tag{7}$$

$$l'_\eta(s) = \#\{k \in \{1, \ldots, i-1\} | \eta_k \geqslant \eta_i\} + \#\{k \in \{i+1, \ldots, n\} | \eta_k > \eta_i\}. \tag{8}$$

Here, for a set A, $\#A$ denotes the number of elements. The quantities $a_\eta(s)$, $a'_\eta(s)$, $l_\eta(s)$ and $l'_\eta(s)$ are called arm length, arm colength, leg length, and leg colength, respectively. Since $l'_\eta(s)$ does not depend on j, we also denote it as $l'_\eta(i)$. Note that for a partition $\lambda \in \Lambda_n^+$, we have

$$l_\lambda(s) = \lambda'_j - i, \tag{9}$$

$$l'_\lambda(s) = i - 1. \tag{10}$$

Further, for a composition $\eta \in \Lambda_n$ and real parameters r and γ, we define the following quantities:

$$f_\eta(r;\gamma) = \prod_{s\in\eta}(a'_\eta(s) - rl'_\eta(s) + \gamma), \tag{11}$$

$$d_\eta(r) = \prod_{s\in\eta}(a_\eta(s) + 1 + r(l_\eta(s) + 1)), \tag{12}$$

$$d'_\eta(r) = \prod_{s\in\eta}(a_\eta(s) + 1 + rl_\eta(s)), \tag{13}$$

$$h_\eta(r) = \prod_{s\in\eta}(a_\eta(s) + r(l_\eta(s) + 1)), \tag{14}$$

$$[0]^r_\eta = \prod_{\substack{s\in\eta\\ s\neq(1,1)}}(a'_\eta(s) - rl'_\eta(s)). \tag{15}$$

2.2. Sutherland model and Jack polynomials

Following [14, 17], we formulate the dynamical spin structure factor $S^{zz}(Q,\omega)$ of the $1/r^2$ supersymmetric *t–J* model based on the freezing technique [15, 16].

As an auxiliary, we introduce the Sutherland model [20–23] with $SU(2,1)$ supersymmetry [24]:

$$\mathcal{H}_{CS} = -\frac{1}{2M}\sum_i \frac{\partial^2}{\partial x_i^2} + \frac{1}{M}\left(\frac{\pi}{L}\right)^2 \sum_{i<j}\frac{\beta(\beta + \tilde{P}_{ij})}{\sin^2\frac{\pi}{L}(x_i - x_j)}. \tag{16}$$

The system has N_{h} holes, $N_\uparrow$ up-spin electrons and $N_\downarrow$ down-spin ones, whose coordinates are represented by x_i^{h} for the ith hole, $x_i^\uparrow$ for the ith up-spin electron and $x_i^\downarrow$ for the ith down-spin electron. We arrange them as $x \equiv (x_1, x_2, \ldots, x_N) = \left(x_1^{\mathrm{h}}, \ldots, x_{N_{\mathrm{h}}}^{\mathrm{h}}, x_1^\downarrow, \ldots, x_{N_\downarrow}^\downarrow, x_1^\uparrow, \ldots, x_{N_\uparrow}^\uparrow\right) \equiv (x^{\mathrm{h}}, x^\downarrow, x^\uparrow)$. Here the graded exchange operator is defined as

$$\tilde{P}_{ij} = \sum_{\alpha,\beta} X_i^{\alpha\beta} X_j^{\beta\alpha}\theta_\beta, \tag{17}$$

where $X_j^{\beta\alpha}$ is the Hubbard operator which changes from state α to β at site j, with α, β being either h (hole state), or one of $\sigma = \uparrow, \downarrow$. The sign factor θ_β is -1 if $\beta = h$ and 1 otherwise. In order to reproduce the lattice model, we take the limit of large β and M, keeping the ratio $t = \beta/M$ fixed. Then the particles crystallize with equal distance from each other, and the resultant dynamics excluding phonons and uniform motion of the centre of gravity is mapped to that of the *t–J* model given by equation (1). It can be shown that the intensity of the phonon correlation is smaller than the spin correlation by a factor of $\mathcal{O}(\beta^{-1})$. Here we take the lattice parameter L/N as the unit of length. Then we have the following relation:

$$\mathcal{H}_{CS} \to t \sum_{i<j} D_{ij}^{-2} \tilde{P}_{ij}. \tag{18}$$

For fixed numbers of $(N_{\rm h}, N_\uparrow, N_\downarrow)$, the right-hand side of the above relation is the t–J model given by equation (1) with a trivial constant shift. Note that the symmetry of the wavefunction leads to the relation $s_{ij}\tilde{P}_{ij} = -1$, where s_{ij} represents the exchange operator of the coordinates of particles i and j. The interval $I = [1, N]$ denotes $\{i \in \mathbf{Z} \mid 1 \leqslant i \leqslant N\}$. We define $I_{\rm h} = [1, N_{\rm h}]$, $I_\downarrow = [N_{\rm h}+1, N_{\rm h}+N_\downarrow]$ and $I_\uparrow = [N_{\rm h}+N_\downarrow+1, N]$. The wavefunction of the ground state for a set of $(N_{\rm h}, N_\downarrow, N_\uparrow)$ is given by

$$\Psi_{\rm GS} = \prod_{i\neq j\in I}\left(1-\frac{z_j}{z_i}\right)^{\beta/2} \prod_{\sigma=\uparrow,\downarrow}\prod_{i\neq j\in I_\sigma}\left(1-\frac{z_j}{z_i}\right)^{1/2}, \tag{19}$$

where the complex coordinates $z = (z_1, \ldots, z_N)$ are related to the original ones $x = (x_1, \ldots, x_N)$ by $z_j = \exp(2\pi {\rm i} x_j/L)$. The spectrum of the Sutherland model is conveniently analysed with the use of a similarity transformation generated by

$$\mathcal{O} = \prod_{i\neq j\in I}\left(1-\frac{z_j}{z_i}\right)^{\beta/2} \prod_{i\in I} z_i^{(N_\uparrow-1)/2}. \tag{20}$$

The transformed Hamiltonian $\hat{\mathcal{H}} = \mathcal{O}^{-1}\mathcal{H}_{\rm CS}\mathcal{O}$ is

$$\hat{\mathcal{H}} = \frac{1}{2M}\left(\frac{2\pi}{L}\right)^2 \sum_{i=1}^{N}\left(\hat{d}_i + \beta\frac{N-1}{2} - \frac{N_\uparrow-1}{2}\right)^2. \tag{21}$$

Here $\hat{d}_i$ is called the Cherednik–Dunkl operator [28, 29] and is given by

$$\hat{d}_i = z_i\frac{\partial}{\partial z_i} + \beta\sum_{k<i}\frac{z_i}{z_i-z_k}(1-s_{ik}) + \beta\sum_{i<k}\frac{z_k}{z_i-z_k}(1-s_{ik}) + \beta(1-i). \tag{22}$$

It is known that $\hat{d}_i$ can be diagonalized simultaneously by homogeneous polynomials. In terms of the monomial $z^\nu = z_1^{\nu_1}\cdots z_N^{\nu_N}$, the resultant eigenfunctions $E_\eta(z;\beta)$ can be expressed as $E_\eta(z;\beta) = z^\eta$ + lower terms (*triangularity*), and are called nonsymmetric Jack polynomials [30, 31]. Here 'lower terms' means a linear combination of the monomial z^ν such that $\nu \prec \eta$. The eigenvalue $\bar{\eta}_i$ of $E_\eta(z;\beta)$ for $\hat{d}_i$ is given by $\bar{\eta}_i = \eta_i - \beta l'_\eta(i)$ for $i = 1, \ldots, N$.

Since we are dealing with identical particles, the eigenfunction should satisfy the following conditions of the $SU(2,1)$ supersymmetry:

(i) symmetric with respect to the exchange between $z_i^{\rm h}$'s;
(ii) antisymmetric with respect to the exchange between z_i^σ's with the same σ.

By taking a linear combination of $E_\lambda(z;\beta)$, we can construct a polynomial $K_\lambda(z;\beta)$ with $SU(2,1)$ supersymmetry [26, 32–34]. The above *triangular* structure of $E_\lambda(z;\beta)$ is inherited to $K_\lambda(z;\beta)$. We specify the set of momenta as $\lambda = (\lambda^{\rm h}, \lambda^\downarrow, \lambda^\uparrow) \in \Lambda_N$, where $\lambda^{\rm h} \in \Lambda^+_{N_{\rm h}}$ and $\lambda^\sigma \in \Lambda^{+,>}_{N_\sigma}$ $(\sigma = \uparrow, \downarrow)$. For the ground state, we have $\lambda = \lambda_{\rm GS} = \left(\lambda^{\rm h}_{\rm GS}, \lambda^\downarrow_{\rm GS}, \lambda^\uparrow_{\rm GS}\right)$ with $\lambda^{\rm h}_{\rm GS} = \left(\frac{N_\uparrow-1}{2}\right)^{N_{\rm h}}$, $\lambda^\downarrow_{\rm GS} = \left(\tilde{\delta}(N_\downarrow) + \left(\frac{N_\uparrow-N_\downarrow}{2}\right)^{N_\downarrow}\right)$ and $\lambda^\uparrow_{\rm GS} = \tilde{\delta}(N_\uparrow)$ (see figure 1). $K_\lambda(z;\beta)$ is normalized so that the coefficient of the monomial z^λ is unity.

We define the inner product of functions $f(z)$ and $g(z)$ in n complex variables, $z = (z_1, z_2, \ldots, z_n)$ as follows:

$$\langle f, g\rangle_n^\beta = \prod_{j=1}^{n}\oint_{|z_j|=1}\frac{{\rm d}z_j}{2\pi{\rm i}z_j}\overline{f(z)}g(z)\prod_{1\leqslant k<l\leqslant n}|z_k-z_l|^{2\beta}, \tag{23}$$

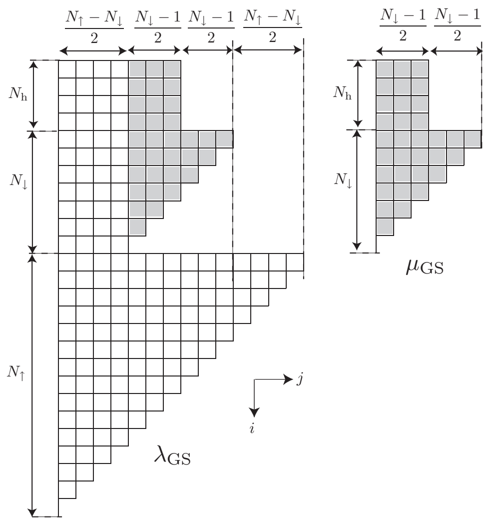

Figure 1. The diagrams of the ground state with $(N, N_\mathrm{h}, N_\downarrow, N_\uparrow) = (26, 4, 7, 15)$. $\lambda_\mathrm{GS} = (4^4, 4^7, 0^{15}) + (3^4, \tilde{\delta}(7), \tilde{\delta}(15))$ (left) and $\mu_\mathrm{GS} = (3^4, \tilde{\delta}(7))$ (right).

where $\overline{f(z)}$ denotes the complex conjugation of $f(z)$. We give some examples of the $SU(2,1)$ Jack polynomials. For the case of $N_\mathrm{h} = N$, the polynomial $K_\lambda(z;\beta)$ $\left(\lambda \in \Lambda_N^+\right)$ reduces to the (monic) Jack polynomial $P_\lambda(z,\beta)$. The antisymmetric Jack polynomial $P_\lambda(z,\beta+1)\prod_{i<j\in I}(z_i - z_j)$ is given by $K_{\lambda+\tilde{\delta}(N)}(z,\beta)$ $\left(\lambda \in \Lambda_N^+\right)$ with $(N_\mathrm{h}, N_\downarrow) = (0,0)$. The $SU(2,1)$ Jack polynomials are orthogonal with respect to the above inner product [34, 35]:

$$\langle K_\lambda, K_\mu\rangle_N^\beta = \delta_{\lambda\mu}\frac{\Gamma[N\beta+1]}{\Gamma[\beta+1]^N}\frac{N_\mathrm{h}!N_\uparrow!N_\downarrow!}{\rho_\lambda(\beta)}\frac{d'_\lambda(\beta)}{d_\lambda(\beta)}\frac{f_\lambda(\beta;1+\beta N)}{f_\lambda(\beta;1+\beta(N-1))}, \tag{24}$$

for compositions $\lambda, \mu \in \Lambda_{N_\mathrm{h}}^+ \times \Lambda_{N_\downarrow}^{+,>} \times \Lambda_{N_\uparrow}^{+,>} \subset \Lambda_N$. Here $\rho_\lambda(\beta)$ is given by the product $\rho_\lambda(\beta) = \rho_\lambda^\mathrm{h}(\beta)\rho_\lambda^\uparrow(\beta)\rho_\lambda^\downarrow(\beta)$ with

$$\rho_\lambda^\mathrm{h}(\beta) = \prod_{i<j\in I_\mathrm{h}}\frac{\bar{\lambda}_i - \bar{\lambda}_j + \beta}{\bar{\lambda}_i - \bar{\lambda}_j}, \tag{25}$$

$$\rho_\lambda^\sigma(\beta) = \prod_{i<j\in I_\sigma}\frac{\bar{\lambda}_i - \bar{\lambda}_j - \beta}{\bar{\lambda}_i - \bar{\lambda}_j} \qquad (\sigma = \uparrow, \downarrow). \tag{26}$$

The operators $n_Q^\mathrm{h} = \sum_j X_j^{00}\,\mathrm{e}^{\mathrm{i}Qj}/\sqrt{N}$ and $n_Q^\sigma = \sum_j X_j^{\sigma\sigma}\,\mathrm{e}^{\mathrm{i}Qj}/\sqrt{N}$ can be expressed for $Q = 2\pi m/N$ as $n_Q^\mathrm{h} = p_m^\mathrm{h}\big/\sqrt{N}$ and $n_Q^\sigma = p_m^\sigma\big/\sqrt{N}$, respectively. Here we have introduced power sums $p_m^\alpha = \sum_{i\in I_\alpha} z_i^m$ $(\alpha = \mathrm{h}, \uparrow, \downarrow)$. In the lattice model we have the completeness relation $\sum_{\sigma=\uparrow,\downarrow} X_i^{\sigma\sigma} + X_i^{00} = 1$. Therefore, in order to calculate $S^{zz}(Q,\omega)$, we need to know two types of the expansion coefficients, c_λ^h and $c_\lambda^\downarrow$, which appear in

$$p_m^\mathrm{h} K_{\lambda_\mathrm{GS}} = \sum_\lambda c_\lambda^\mathrm{h} K_\lambda(z;\beta), \tag{27}$$

$$p_m^\downarrow K_{\lambda_\mathrm{GS}} = \sum_\lambda c_\lambda^\downarrow K_\lambda(z;\beta). \tag{28}$$

Using these coefficients, in the lattice limit $(\beta \to \infty)$, the spin operator S_j^z can be expressed by $S_j^z = \left(X_j^{\uparrow\uparrow} - X_j^{\downarrow\downarrow}\right)\big/2 = 1/2 - X_j^{00}\big/2 - X_j^{\downarrow\downarrow}$. Therefore for $Q > 0$, we have the following

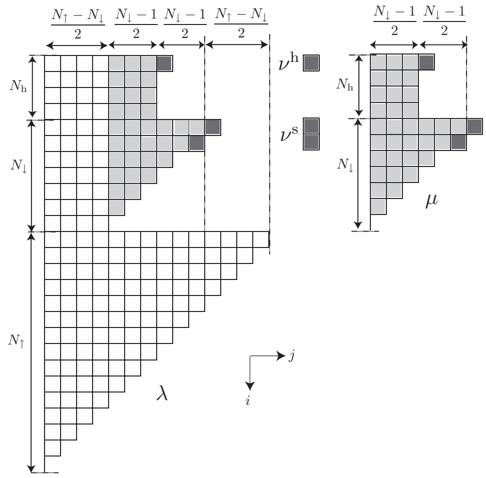

Figure 2. The diagrams that appear in the expansions (27) and (28) under the conditions (30) and (31) (the small momentum region). These diagrams correspond to the case with $(N, N_\mathrm{h}, N_\downarrow, N_\uparrow) = (26, 4, 7, 15)$. $\lambda = \lambda_{\mathrm{GS}} + (\nu^\mathrm{h}, \nu^\mathrm{s}, 0^{N_\uparrow})$ (left) and $\mu = \mu_{\mathrm{GS}} + (\nu^\mathrm{h}, \nu^\mathrm{s})$ (right) with $\nu^\mathrm{h} = (1, 0^3)$ and $\nu^\mathrm{s} = (1^2, 0^5)$. The diagrams λ_{GS} and μ_{GS} are shown in figure 1. The diagrams shown in this figure do not contribute to $S^{zz}(Q, \omega)$, owing to $\nu^\mathrm{h} = (1, 0^{N_\mathrm{h}-1})$ (see equation (45)).

relation:

$$S^{zz}(Q, \omega) = \frac{1}{N} \sum_{\lambda} \left(c_\lambda^\downarrow + \frac{c_\lambda^\mathrm{h}}{2} \right)^2 \frac{\langle K_\lambda, K_\lambda \rangle_N}{\langle K_{\lambda_{\mathrm{GS}}}, K_{\lambda_{\mathrm{GS}}} \rangle_N} \delta(\omega - \Delta E_\lambda). \tag{29}$$

For $Q = 0$, there is a finite intensity only at $\omega = 0$, which is given by $N\bar{m}^2/4$. It is difficult to derive c_λ^h and $c_\lambda^\downarrow$ for general values of $Q = 2\pi m/N$. In the next section, we show that there occurs a drastic simplification in the small momentum region.

3. Matrix element

In the small momentum region, we have some special properties that the calculation for the expansion coefficient and norm can be essentially reduced to those of symmetric Jack polynomials [27]. First we summarize the necessary formula to evaluate the expansion coefficients $c_\lambda^\downarrow$ and c_λ^h. Next we obtain the expansion coefficients by the use of the replica type technique [19].

3.1. Small momentum region

We consider the case where the following two conditions are satisfied:

$$m \leqslant (N_\uparrow - N_\downarrow)/2, \tag{30}$$

$$m \leqslant (N_\downarrow - 1)/2. \tag{31}$$

(Note that these conditions constitute the small momentum region.) In this region, owing to the *triangular* structure of the polynomial $K_\lambda(z, \beta)$, the composition λ contributing to the summation (29) is restricted to the form $\lambda = \lambda_{\mathrm{GS}} + (\nu^\mathrm{h}, \nu^\mathrm{s}, 0^{N_\uparrow})$ (see figures 1 and 2). We define $\mu_{\mathrm{GS}} = \left(\mu^{\mathrm{h}}_{\mathrm{GS}}, \mu^{\downarrow}_{\mathrm{GS}}\right)$ with $\mu^{\mathrm{h}}_{\mathrm{GS}} = \left(\left(\frac{N_\downarrow - 1}{2}\right)^{N_{\mathrm{h}}}\right)$ and $\mu^{\downarrow}_{\mathrm{GS}} = \tilde{\delta}(N_\downarrow)$. Under conditions (30) and (31), by extending a calculation by Baker and Forrester [32], we obtain the following relation:

$$K_\lambda(z;\beta) = \tilde{K}_\mu(\tilde{z};\beta') \prod_{j\in I_{\mathrm{h}}\cup I_\downarrow} z_j^{\frac{N_\uparrow - N_\downarrow}{2}} \prod_{i<j\in I_\uparrow} (z_i - z_j), \tag{32}$$

where $\beta' = \beta/(\beta+1)$, $\mu = \left(\mu^{\mathrm{h}}_{\mathrm{GS}}, \mu^{\downarrow}_{\mathrm{GS}}\right) + (\nu^{\mathrm{h}}, \nu^{\mathrm{s}}) \in \Lambda^{+}_{N_{\mathrm{h}}} \times \Lambda^{+,>}_{N_\downarrow}$ and $\tilde{z} = (z^{\mathrm{h}}, z^{\downarrow})$. Here $\tilde{K}_\mu(\tilde{z};\beta')$ is a Jack polynomial with $SU(1,1)$ supersymmetry, which is a linear combination of $N_{\mathrm{h}} + N_\downarrow$ variables non-symmetric Jack polynomials $E_\eta(\tilde{z};\beta')$. The polynomial $E_\eta(\tilde{z};\beta')$ can be obtained by substitution $z_j = 0$ for $j \in [N_{\mathrm{h}} + N_\downarrow + 1, N]$ in N variables non-symmetric Jack polynomial $E_\eta(z;\beta')$. The $SU(1,1)$ Jack polynomial $\tilde{K}_\mu(\tilde{z};\beta')$ is symmetric with respect to the exchange between z_i^{h}'s and antisymmetric with respect to the exchange between $z_i^{\downarrow}$'s. For the composition μ_{GS}, the $SU(1,1)$ Jack polynomial $\tilde{K}_{\mu_{\mathrm{GS}}}(\tilde{z},\beta')$ is independent of the parameter β', and is explicitly given by

$$\tilde{K}_{\mu_{\mathrm{GS}}}(\tilde{z},\beta') = \prod_{j\in I_{\mathrm{h}}} z_j^{(N_\downarrow - 1)/2} \prod_{i<j\in I_\downarrow} (z_i - z_j). \tag{33}$$

Under conditions (30) and (31), the norm of the above states can be reduced to the following form [27]:

$$\frac{\langle K_\lambda, K_\lambda\rangle^{\beta}_{N}}{\left\langle K_{\lambda_{\mathrm{GS}}}, K_{\lambda_{\mathrm{GS}}}\right\rangle^{\beta}_{N}} = \frac{\langle \tilde{K}_\mu, \tilde{K}_\mu\rangle^{\beta'}_{N_{\mathrm{h}}+N_\downarrow}}{\left\langle \tilde{K}_{\mu_{\mathrm{GS}}}, \tilde{K}_{\mu_{\mathrm{GS}}}\right\rangle^{\beta'}_{N_{\mathrm{h}}+N_\downarrow}} = \frac{d'_{\nu^{\mathrm{h}}}(\beta'')}{h_{\nu^{\mathrm{h}}}(\beta'')} \frac{f_{\nu^{\mathrm{h}}}(\beta'';\beta'' N_{\mathrm{h}})}{f_{\nu^{\mathrm{h}}}(\beta'';1+\beta''(N_{\mathrm{h}}-1))}$$
$$\times \frac{d'_{\nu^{\mathrm{s}}}(\bar{\beta}')}{h_{\nu^{\mathrm{s}}}(\bar{\beta}')} \frac{f_{\nu^{\mathrm{s}}}(\bar{\beta}';\bar{\beta}'(N_\downarrow + \beta'' N_{\mathrm{h}}))}{f_{\nu^{\mathrm{s}}}(\bar{\beta}';1+\bar{\beta}'(N_\downarrow + \beta'' N_{\mathrm{h}} - 1))}, \tag{34}$$

where $\beta'' = \beta'/(\beta'+1) = \beta/(2\beta+1)$ and $\bar{\beta}' = \beta' + 1 = (2\beta+1)/(\beta+1)$. We remark that for monic symmetric Jack polynomials $P_\lambda(z,\beta)$, the following relation is obtained [25]:

$$\frac{\langle P_\lambda, P_\lambda\rangle^{\beta}_{N}}{\langle 1, 1\rangle^{\beta}_{N}} = \frac{d'_\lambda(\beta)}{h_\lambda(\beta)} \frac{f_\lambda(\beta;\beta N)}{f_\lambda(\beta;1+\beta(N-1))}. \tag{35}$$

Relations (32) and (34) hold if both the conditions $0 \leqslant \lambda_j \leqslant N_\uparrow - 1$ for $j \in I_{\mathrm{h}} \cup I_\downarrow$ and $\lambda^\uparrow = \lambda^\uparrow_{\mathrm{GS}}$ are satisfied. However if equations (30) and (31) are not satisfied, the K_λ without satisfying equation (32) may involve in the summations (27) and (28). In the strong coupling limit ($\beta \to \infty$), the parameter $\beta' = \beta/(\beta+1)$ approaches unity. In this limit, the eigenstates satisfying the above conditions '$0 \leqslant \lambda_j \leqslant N_\uparrow - 1$ for $j \in I_{\mathrm{h}} \cup I_\downarrow$ and $\lambda^\uparrow = \lambda^\uparrow_{\mathrm{GS}}$' are $SU(2,1)$ Yangian highest weight states (YHWS) [36] in the $1/r^2$ supersymmetric *t–J* model, which can be mapped to that of the $SU(1,1)$ Sutherland model with coupling parameter $\beta = 1$ [37]. As employed in [27, 38, 39], if the excited states are restricted within the YHWS, the derivation of the correlation functions can be reduced to that for the Sutherland model with $SU(1,1)$ supersymmetry. We would like to stress that if the small momentum conditions (30) and (31) are not satisfied, non-YHWS may contribute in the excited states of $S^{zz}(Q,\omega)$.

3.2. Replica type technique

Using the replica type technique [19], we derive the analytic formula of the coefficients c^{h}_λ and $c^{\downarrow}_\lambda$. For indices $\alpha = \mathrm{h}$ and $\downarrow$, we define the quantity $\mathcal{Z}^\alpha(\theta)$ as follows:

$$\mathcal{Z}^\alpha(\theta) = \prod_{j\in I_\alpha} (1 - \mathrm{e}^{-\mathrm{i}\theta} z_j). \tag{36}$$

For any real parameter u, we have the following relation:

$$(\mathcal{Z}^\alpha(\theta))^u = \exp\left(u\sum_{j\in I_\alpha}\ln(1-\mathrm{e}^{-\mathrm{i}\theta}z_j)\right) = \exp\left(-u\sum_{m=1}^{\infty}\mathrm{e}^{-\mathrm{i}m\theta}\frac{p_m^\alpha}{m}\right).$$

By use of the above relation, the expansion coefficients c_λ^α ($\alpha = \mathrm{h}, \downarrow$) in equations (27) and (28) are given by

$$c_\lambda^\alpha = \frac{\langle K_\lambda, (p_m^\alpha \times K_{\lambda_{\mathrm{GS}}})\rangle_N^\beta}{\langle K_\lambda, K_\lambda\rangle_N^\beta} = \lim_{u\to 0}\frac{1}{\mathrm{i}}\frac{1}{u}\frac{\partial}{\partial\theta}\left.\frac{\langle K_\lambda, (\mathcal{Z}^\alpha(\theta))^u K_{\lambda_{\mathrm{GS}}}\rangle_N^\beta}{\langle K_\lambda, K_\lambda\rangle_N^\beta}\right|_{\theta=0}, \tag{37}$$

where momentum conservation $\|\lambda\| - \|\lambda_{\mathrm{GS}}\| = m$ is satisfied. We introduce the similar type of expansions for the $SU(1,1)$ Jack polynomials:

$$p_m^{\mathrm{h}}\tilde{K}_{\mu_{\mathrm{GS}}}(\tilde{z}, \beta') = \sum_\mu \tilde{c}_\mu^{\mathrm{h}}\tilde{K}_\mu(\tilde{z}, \beta'), \tag{38}$$

$$p_m^{\downarrow}\tilde{K}_{\mu_{\mathrm{GS}}}(\tilde{z}, \beta') = \sum_\mu \tilde{c}_\mu^{\downarrow}\tilde{K}_\mu(\tilde{z}, \beta'). \tag{39}$$

In a similar manner, the expansion coefficients of $\tilde{c}_\mu^\alpha$ ($\alpha = \mathrm{h}, \downarrow$) in equations (38) and (39) are given by

$$\tilde{c}_\mu^\alpha = \lim_{u\to 0}\frac{1}{\mathrm{i}}\frac{1}{u}\frac{\partial}{\partial\theta}\left.\frac{\langle \tilde{K}_\mu, (\mathcal{Z}^\alpha(\theta))^u\tilde{K}_{\mu_{\mathrm{GS}}}\rangle_{N_{\mathrm{h}}+N_\downarrow}^{\beta'}}{\langle \tilde{K}_\mu, \tilde{K}_\mu\rangle_{N_{\mathrm{h}}+N_\downarrow}^{\beta'}}\right|_{\theta=0}, \tag{40}$$

where $\|\mu\| - \|\mu_{\mathrm{GS}}\| = m$.

Under conditions (30) and (31), by use of equation (32), for given compositions $\lambda = \lambda_{\mathrm{GS}} + (\nu^{\mathrm{h}}, \nu^{\mathrm{s}}, 0^{N_\uparrow})$ and $\mu = \mu_{\mathrm{GS}} + (\nu^{\mathrm{h}}, \nu^{\mathrm{s}})$, [4] we can show the relation[5]

$$c_\lambda^\alpha = \tilde{c}_\mu^\alpha \qquad (\alpha = \mathrm{h}, \downarrow). \tag{41}$$

This relation means that the expansion coefficients for the $SU(2,1)$ Jack polynomials can be expressed by those for the $SU(1,1)$ Jack polynomials, provided the conditions (30) and (31) are satisfied.

Next, for given parameters (p, q), we consider the following expansion:

$$(\mathcal{Z}^{\mathrm{h}}(0))^p(\mathcal{Z}^{\downarrow}(0))^q\tilde{K}_{\mu_{\mathrm{GS}}}(\tilde{z}, \beta') = \sum_\mu \chi_\mu(\beta')\tilde{K}_\mu(\tilde{z}; \beta'). \tag{42}$$

The formula of the expansion coefficient $\chi_\mu(\beta')$ for arbitrary μ has not been proved yet. However, in the small momentum region, we have a formula for $\chi_\mu(\beta')$. For $\mu = \mu_{\mathrm{GS}} + (\nu^{\mathrm{h}}, \nu^{\mathrm{s}})$, we have the following relation [27]:

$$\chi_\mu(\beta') = \frac{f_{\nu^{\mathrm{h}}}(\beta''; -p+\beta' q)}{d'_{\nu^{\mathrm{h}}}(\beta'')} \times \frac{f_{\nu^{\mathrm{s}}}(\bar{\beta}'; -q)}{d'_{\nu^{\mathrm{s}}}(\bar{\beta}')}. \tag{43}$$

By use of equations (40) and (43) with $(p, q) = (0, u)$, we can obtain the coefficient $c_\lambda^\downarrow$. For $\lambda = \lambda_{\mathrm{GS}} + (\nu^{\mathrm{h}}, \nu^{\mathrm{s}}, 0^{N_\uparrow})$, it is given by

$$c_\lambda^\downarrow = -\beta'' m\frac{[0]_{\nu^{\mathrm{h}}}^{\beta''} \times f_{\nu^{\mathrm{s}}}(\bar{\beta}'; 0)}{d'_{\nu^{\mathrm{h}}}(\beta'')d'_{\nu^{\mathrm{s}}}(\bar{\beta}')} + m\frac{f_{\nu^{\mathrm{h}}}(\beta''; 0) \times [0]_{\nu^{\mathrm{s}}}^{\bar{\beta}'}}{d'_{\nu^{\mathrm{h}}}(\beta'')d'_{\nu^{\mathrm{s}}}(\bar{\beta}')}, \tag{44}$$

[4] The ν^{h} and ν^{s} are the partitions with $\nu^{\mathrm{h}} \in \Lambda_{N_{\mathrm{h}}}^+$ and $\nu^{\mathrm{s}} \in \Lambda_{N_\downarrow}^+$, respectively.

[5] In the strong coupling limit $\beta \to \infty$, we have $\beta' \to 1$. For YHWS, equations (34) and (41) reflect the equivalence between the freezing approach and mapping of the eigenstates of the $1/r^2$ supersymmetric t–J model into those of the $SU(1,1)$ Sutherland model.

where $m = \|\nu^{\rm h}\| + \|\nu^{\rm s}\|$. In a similar manner, we can derive the coefficient $c^{\rm h}_\lambda$ (see equation (A.3)). The first term in equation (44) can then be represented by $-\beta'' c^{\rm h}_\lambda$, which vanishes unless $\nu^{\rm s} = (0^{N_\downarrow})$. We consider the quantity $c_\lambda = c^{\downarrow}_\lambda + \beta'' c^{\rm h}_\lambda$, which is nothing but the second term of the right-hand side in equation (44). Owing to the factor $f_{\nu^{\rm h}}(\beta''; 0)$, the c_λ vanishes unless $\nu^{\rm h} = (0^{N_{\rm h}})$. Therefore, the c_λ can be simplified as

$$c_\lambda = \|\nu^{\rm s}\| \frac{[0]^{\bar\beta'}_{\nu^{\rm s}}}{d'_{\nu^{\rm s}}(\bar\beta')} \times \delta_{(\nu^{\rm h}),(0^{N_{\rm h}})}. \tag{45}$$

We notice that while $c^{\downarrow}_\lambda$ is related to the expansion coefficients for the $SU(1,1)$ Jack polynomials, the quantity c_λ given by equation (45) is related to those for the monic symmetric Jack polynomials. In fact, for the monic symmetric Jack polynomials, one has the following formula [40]:

$$\sum_{j=1}^{n} z_j^m = m \times \sum_{\substack{\lambda\in\Lambda_n^+ \\ \|\lambda\|=m}} \frac{[0]^{\beta}_{\lambda}}{d'_{\lambda}(\beta)} P_\lambda(z;\beta). \tag{46}$$

In the strong coupling limit ($\beta \to \infty$), we have $\beta'' \to 1/2$, and therefore the c_λ approaches $c^{\downarrow}_\lambda + c^{\rm h}_\lambda/2$. This is precisely the quantity which appears in $S^{zz}(Q,\omega)$ (see equation (29)). Thus, no charge excitation contributes to $S^{zz}(Q,\omega)$ in the small momentum region. Since we have $\bar\beta' \to 2$ in the limit $\beta \to \infty$, the coefficient c_λ vanishes owing to the factor $[0]^{\bar\beta'}_{\nu^{\rm s}}$, if partition $\nu^{\rm s}$ contains $s = (i, j) = (2, 3)$. Then, $S^{zz}(Q,\omega)$ is determined by three parameters $(\lambda_{\rm a}, \lambda_{\rm s1}, \lambda_{\rm s2})$. Partition $\nu^{\rm s}$ is restricted to $\nu^{\rm s} = (\lambda_{\rm a}, 2^{\lambda_{\rm s2}-1}, 1^{\lambda_{\rm s1}-\lambda_{\rm s2}}, 0^{N_\downarrow-\lambda_{\rm s1}})$. These three parameters are related directly with momenta of the elementary excitations: one antispinon and two spinons.

4. Results

Using relations (34) and (45), we can express $S^{zz}(Q,\omega)$ in terms of three parameters $(\lambda_{\rm a}, \lambda_{\rm s1}, \lambda_{\rm s2})$. In the small momentum region (see equations (30) and (31)), we obtain the following expression:

$$\begin{aligned} S^{zz}(Q,\omega) = \frac{m^2}{2N} \sum_{\lambda_{\rm s1}\geqslant\lambda_{\rm s2},\lambda_{\rm a}} & \delta_{m,\|\nu^{\rm s}\|}\delta(\omega-\Delta E_\lambda)(\lambda_{\rm a}-1)(\lambda_{\rm a}+N-(N_\uparrow-N_\downarrow)-1) \\ & \times\left(\lambda_{\rm s1}-\lambda_{\rm s2}+\frac12\right)\prod_{j=1}^{2}\frac{1}{(\lambda_{\rm a}+2\lambda_{{\rm s}j}-j-1)(\lambda_{\rm a}+2\lambda_{{\rm s}j}-j)} \\ & \times\prod_{j=1}^{2}\frac{\Gamma\left[\lambda_{{\rm s}j}-\frac{j-1}{2}\right]\Gamma\left[\frac N2-\frac{N_\uparrow-N_\downarrow}{2}-\lambda_{{\rm s}j}+\frac j2\right]}{\Gamma\left[\lambda_{{\rm s}j}-\frac{j-2}{2}\right]\Gamma\left[\frac N2-\frac{N_\uparrow-N_\downarrow}{2}-\lambda_{{\rm s}j}+\frac{j+1}{2}\right]}, \end{aligned} \tag{47}$$

where $Q = 2\pi m/N$ and $\|\nu^{\rm s}\| = \lambda_{\rm a} + \sum_{j=1}^{2}(\lambda_{{\rm s}j}-1)$. The excitation energy ΔE_λ is given by

$$\begin{aligned} \Delta E_\lambda/t = \left(\frac{2\pi}{N}\right)^2 \Bigg[& (\lambda_{\rm a}-2)\left(\frac N2-\frac{N_\uparrow-N_\downarrow}{2}+\frac{\lambda_{\rm a}}{2}\right) \\ & + \sum_{j=1}^{2}\lambda_{{\rm s}j}\left(\frac N2-\frac{N_\uparrow-N_\downarrow}{2}-\lambda_{{\rm s}j}+j-\frac12\right)\Bigg]. \end{aligned} \tag{48}$$

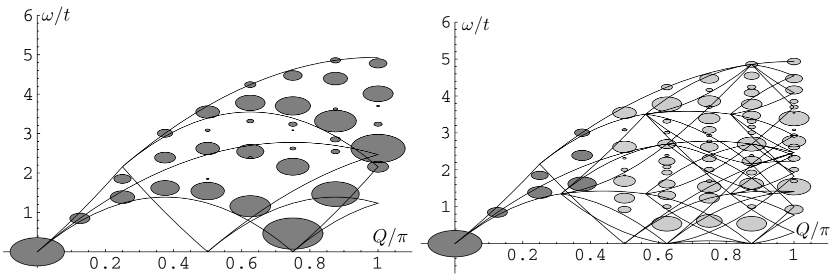

Figure 3. $S^{zz}(Q,\omega)$ for $(N, N_{\rm h}, N_\uparrow, N_\downarrow) = (16, 0, 10, 6)$ (left) and $(16, 2, 9, 5)$ (right). Each spectral weight is proportional to the area of the oval. For dark shaded ovals in the right figure, excitation energies and spectral weights agree with those in the Haldane–Shastry model (the left figure), which can be expressed by equations (47) and (48).

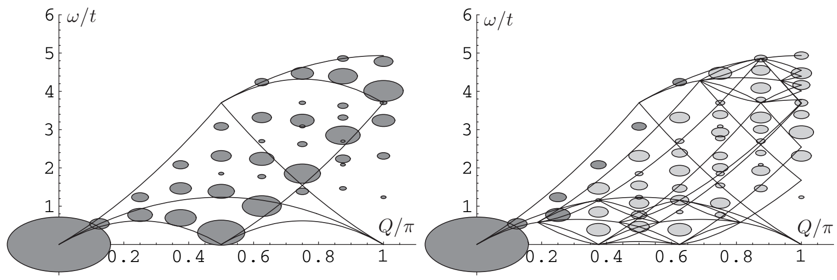

Figure 4. $S^{zz}(Q,\omega)$ for $(N, N_{\rm h}, N_\uparrow, N_\downarrow) = (16, 0, 12, 4)$ (left) and $(16, 2, 11, 3)$ (right). For dark shaded ovals in the right figure, excitation energies and spectral weights agree with those in the Haldane–Shastry model (the left figure), which can be expressed by equations (47) and (48).

In the case of $\lambda_{\rm s2} = 0$, where only one spinon is excited, we need to modify the above results as follows:

$$S^{zz}(Q,\omega) = \frac{1}{2^2 N}\frac{\Gamma\left[\frac{1}{2}\right]\Gamma[\lambda_{\rm s1}+1]\Gamma\left[\frac{N}{2}-\frac{N_\uparrow - N_\downarrow}{2}+1\right]\Gamma\left[\frac{N}{2}-\frac{N_\uparrow - N_\downarrow}{2}-\lambda_{\rm s1}+\frac{1}{2}\right]}{\Gamma\left[\lambda_{\rm s1}+\frac{1}{2}\right]\Gamma\left[\frac{N}{2}-\frac{N_\uparrow - N_\downarrow}{2}+\frac{1}{2}\right]\Gamma\left[\frac{N}{2}-\frac{N_\uparrow - N_\downarrow}{2}-\lambda_{\rm s1}+1\right]}\delta(\omega - \Delta E_\lambda), \tag{49}$$

where $Q = 2\pi\lambda_{\rm s1}/N$. The excitation energy ΔE_λ is given by $(\lambda_{\rm s1}, \lambda_{\rm s2}, \lambda_{\rm a}) = (\lambda_{s1}, 0, 2)$ in equation (48). We have checked the validity by comparison with numerical results up to $N = 16$ [12] (see appendix B). In figures 3 and 4, we show the results for $N = 16$. From comparison with numerical results, the analytic expression of the two-spinon plus one-antispinon contribution can be applied in the wider range of (Q,ω) (see appendix B). The analytic expressions (47)–(49) coincide with those for the Haldane–Shastry model [14]. From the above results for finite systems, we can derive the analytic expression of $S^{zz}(Q,\omega)$ in the thermodynamic limit (see equation (4)). Note that the contribution in the case of $\lambda_{\rm s2} = 0$ vanishes in the thermodynamic limit. Thus we have proved analytically that in the momentum region $0 < Q \leqslant \min[\pi\bar{m}, k_{{\rm F},\downarrow}]$, the structure factor $S^{zz}(Q,\omega)$ is not affected by hole doping. Here $k_{{\rm F},\sigma}$ is given by $\pi\bar{n}_\sigma$. In this region, $S^{zz}(Q,\omega)$ diverges as $(\omega - \epsilon_{\rm s}(Q))^{-1/2}$, as the frequency approaches the lower edge corresponding to the spinon dispersion $\omega = \epsilon_{\rm s}(Q)$. The obtained $S^{zz}(Q,\omega)$ has the same form as the dynamical density–density correlation function of the spinless Sutherland model with coupling $\beta = 2$ except the momentum range [41, 42]. For

$0 < Q \leqslant \min[\pi\bar{m}, k_{\mathrm{F},\downarrow}]$, the static structure factor $S^{zz}(Q)$ can be evaluated by the integration $\int \mathrm{d}\omega\, S^{zz}(Q,\omega)$, which reproduces the expressions presented in appendix C.

5. Summary

By use of the freezing technique on the Sutherland model with $SU(2,1)$ supersymmetry and the replica type technique, we have obtained the dynamical spin structure factor $S^{zz}(Q,\omega)$ with $Q \leqslant \min[\pi\bar{m}, k_{\mathrm{F},\downarrow}]$ in the supersymmetric t–J model with $1/r^2$ interaction. The $S^{zz}(Q,\omega)$ has the same form as that of the Haldane–Shastry model in this small momentum region. In the thermodynamic limit, two spinons and one antispinon contribute to $S^{zz}(Q,\omega)$. Therefore, $S^{zz}(Q,\omega)$ is not affected by hole doping in this region. Thus we have proved the strong spin–charge separation in $S^{zz}(Q,\omega)$, which was numerically obtained in the previous paper [12]. From comparison with numerical results, we have found that the analytic expression of two-spinon plus one-antispinon contribution can be applied to the wider range of (Q,ω).

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Appendix A. Dynamical charge structure factor $N(Q,\omega)$

We derive the dynamical charge structure factor $N(Q,\omega)$ for $Q \leqslant k_{F,\downarrow}$ [6] by use of the replica type technique [19]. The $N(Q,\omega)$ is defined by

$$N(Q,\omega) = \sum_{\nu} |\langle \nu | n_Q | 0 \rangle|^2 \delta(\omega - E_\nu + E_0), \tag{A.1}$$

where $n_Q = \sum_l n_l\, \mathrm{e}^{\mathrm{i}Ql}/\sqrt{N}$. As in our previous study [17], using the expansion coefficient c_λ^{h} in equation (27), the dynamical charge structure factor $N(Q,\omega)$ can be expressed as

$$N(Q,\omega) = \frac{1}{N} \sum_{\lambda} \left(c_\lambda^{\mathrm{h}}\right)^2 \frac{\langle K_\lambda, K_\lambda \rangle_N}{\langle K_{\lambda_{\mathrm{GS}}}, K_{\lambda_{\mathrm{GS}}} \rangle_N} \delta(\omega - \Delta E_\lambda). \tag{A.2}$$

for the momentum $Q > 0$. Using the replica type technique [19], we can obtain the coefficient c_λ^{h}. By use of equations (40) and (43) with $(p,q) = (u,0)$, the expansion coefficient c_λ^{h} can be derived as

$$c_\lambda^{\mathrm{h}} = m \frac{[0]_{\nu^{\mathrm{h}}}^{\beta''} \times f_{\nu^{\mathrm{s}}}(\bar{\beta}'; 0)}{d'_{\nu^{\mathrm{h}}}(\beta'') d'_{\nu^{\mathrm{s}}}(\bar{\beta}')} = m \frac{[0]_{\nu^{\mathrm{h}}}^{\beta''}}{d'_{\nu^{\mathrm{h}}}(\beta'')} \delta_{(\nu^{\mathrm{s}}),(0^{N_\downarrow})}, \tag{A.3}$$

in the small momentum region $m \leqslant (N_\downarrow - 1)/2$. Here we have used the property that $f_{\nu^{\mathrm{s}}}(\bar{\beta}'; 0)$ vanishes unless $\nu^{\mathrm{s}} = (0^{N_\downarrow})$. In fact, as shown in [17], the excited states contributing to $N(Q,\omega)$ in this small momentum region are restricted to the case where $\lambda = \lambda_{\mathrm{GS}} + (\nu^{\mathrm{h}}, 0^{N_\downarrow}, 0^{N_\uparrow})$. In this case, the $SU(2,1)$ Jack polynomial $K_\lambda(z,\beta)$ in equation (27) has a form $K_\lambda(z,\beta) = P_{\nu^{\mathrm{h}}}(z^{\mathrm{h}}, \beta'') \times K_{\lambda_{\mathrm{GS}}}$. Therefore the coefficient c_λ^{h} can be derived via equation (46) as well. Namely, the use of the replica type technique is not essential for the derivation of c_λ^{h} in contrast to the case for $c_\lambda^{\downarrow}$. The norm can be evaluated by the reduced formula equation (34). In the strong coupling limit $\beta \to \infty$, the coupling parameter

[6] For the derivation of $N(Q,\omega)$, the condition $Q \leqslant \pi\bar{m}$, i.e., equation (30) is not necessary.

Table B1. Comparison between analytic results and numerical ones [12] for $(N, N_h, N_\uparrow, N_\downarrow) = (16, 2, 9, 5)$.

$(\lambda_{s1}, \lambda_{s2}, \lambda_a)$	q/π	ω/t	i_λ (analytic)	i_λ (numeric)
(1, 0, 1)	$\frac{1}{8}$	$\frac{11\pi^2}{128}$ ($\simeq$0.848 169)	$\frac{3}{88}$ ($\simeq$0.034 090 90)	0.034 090 80
(2, 0, 1)	$\frac{1}{4}$	$\frac{9\pi^2}{64}$ ($\simeq$1.387 91)	$\frac{5}{99}$ ($\simeq$0.050 5050)	0.050 504 42
(1, 1, 2)		$\frac{3\pi^2}{16}$ ($\simeq$1.850 55)	$\frac{13}{528}$ ($\simeq$0.024 621 21)	0.024 621 07
(3, 0, 1)	$\frac{3}{8}$	$\frac{21\pi^2}{128}$ ($\simeq$1.619 23)	$\frac{16}{231}$ ($\simeq$0.069 264 06)	0.069 263 59
(2, 1, 2)		$\frac{31\pi^2}{128}$ ($\simeq$2.390 29)	$\frac{13}{352}$ ($\simeq$0.036 931 818)	0.036 931 62
(1, 1, 3)		$\frac{39\pi^2}{128}$ ($\simeq$3.007 15)	$\frac{7}{352}$ ($\simeq$0.019 886 36)	0.019 886 34
	$\frac{1}{2}$	...	...	
(2, 2, 2)		$\frac{5\pi^2}{16}$ ($\simeq$3.084 25) ...	$\frac{13}{6480}$ ($\simeq$0.002 006 17) ...	0.002 006 48

β'' becomes $1/2$. In the thermodynamic limit, the dynamical charge structure factor $N(Q,\omega)$ for $0 < Q \leqslant k_{F,\downarrow}$ can be expressed as [17]

$$N(Q,\omega) = \frac{Q^2}{\pi}\int_0^{2\pi-4k_F} dq_{ah}\int_0^{k_{F,\downarrow}} dq_1\int_0^{k_{F,\downarrow}} dq_2\,\delta\left(Q - q_{ah} - \sum_{j=1}^{2} q_j\right) \times\delta\left(\omega - \epsilon_{ah}(q_{ah}) - \sum_{j=1}^{2}\epsilon_h(q_j)\right)\frac{|q_1-q_2|\epsilon_{ah}(q_{ah})}{\prod_{j=1}^{2}\sqrt{\epsilon_h(q_j)}(2q_j+q_{ah})^2}, \tag{A.4}$$

where the Fermi momentum k_F is given by $k_F = \pi\bar{n}/2$, $\epsilon_h(q)$ is the holon spectrum: $\epsilon_h(q) = q(v_c + q)$ and $\epsilon_{ah}(q)$ is the antiholon spectrum: $\epsilon_{ah}(q) = q(v_c - q/2)$. Here the charge velocity v_c is $v_c = \pi(1-\bar{n})$. This expression has the same form as the dynamical density–density correlation function of the spinless Sutherland model with coupling parameter $\beta = 1/2$.

Appendix B. Comparison with numerical results

We make a comparison between analytic results and numerical ones in $S^{zz}(Q,\omega)$ [12]. In tables B1 and B2, we present the cases $(N, N_h, N_\uparrow, N_\downarrow) = (16, 2, 9, 5)$ and $(16, 2, 11, 3)$, respectively. Our analytic proof is restricted to the case where $Q \leqslant \min[k_{F,\downarrow}, \pi\bar{m}]$. However, the analytical expression of the two-spinon plus one-antispinon contribution can be applied in the wider range. As a result of hole doping, the integration ranges of the spinon momenta in equation (4) are changed to $0 < q_i < k_{F,\downarrow}$ for $i = 1$ and 2. From comparison with numerical results [10, 12], we find that a similar fact occurs also in the $N(Q,\omega)$. Namely, although analytic derivation of $N(Q,\omega)$ is restricted to the region $0 < Q < k_{F,\downarrow}$, the expression of the (right-moving) two-holon plus one-antiholon contribution can be extended to the integration range shown in equation (A.4).

Appendix C. Static structure factors

We consider the static structure factors $S^{zz}(Q)$ and $N(Q)$. There are several ways to obtain these quantities. If one knows the dynamical structure factors $S^{zz}(Q,\omega)$ and $N(Q,\omega)$, then the

Table B2. Comparison between analytic results and numerical ones [12] for $(N, N_h, N_\uparrow, N_\downarrow) = (16, 2, 11, 3)$.

$(\lambda_{s1}, \lambda_{s2}, \lambda_a)$	Q/π	ω/t	I_λ (analytic)	I_λ (numeric)
(1, 0, 1)	$\frac{1}{8}$	$\frac{7\pi^2}{128}$ ($\simeq$0.539 744)	$\frac{1}{28}$ ($\simeq$0.035 714 28)	0.035 714 172
(2, 0, 1)	$\frac{1}{4}$	$\frac{5\pi^2}{64}$ ($\simeq$0.771 063)	$\frac{2}{35}$ ($\simeq$0.057 142 857)	0.057 142 144
(1, 1, 2)		$\frac{\pi^2}{8}$ ($\simeq$1.2337)	$\frac{3}{112}$ ($\simeq$0.026 785 71)	0.026 785 570
	$\frac{3}{8}$	...	...	
(1, 1, 3)		$\frac{27\pi^2}{128}$ ($\simeq$2.081 87)	$\frac{5}{224}$ ($\simeq$0.022 321 428)	0.022 321 422
	$\frac{1}{2}$	...	...	
(1, 1, 4)		$\frac{5\pi^2}{16}$ ($\simeq$3.084 25)	$\frac{11}{560}$ ($\simeq$0.019 642 857)	0.019 642 881
	$\frac{5}{8}$	...	...	
(1, 1, 5)		$\frac{55\pi^2}{128}$ ($\simeq$4.240 85)	$\frac{1}{56}$ ($\simeq$0.017 857 14)	0.017 857 178
		...	...	

static structure factors can be obtained by integration over ω. Gebhard and Vollhardt calculated the static structure factors for $\bar{m} = 0$, from the Gutzwiller wavefunction [43]. For general $\bar{m}$, Forrester derived the analytic expressions of the equal-time two-point correlation functions [44, 45][7]. In the following, we obtain the static structure factors by Fourier transformation of these equal-time two-point correlation functions.

The equal-time two-point correlation functions are defined by $C^{zz}(x) \equiv \langle 0|S_x^z S_0^z|0\rangle$ and $C^{hh}(x) \equiv \langle 0|n_x n_0|0\rangle$. They are expressed as follows,

$$C^{zz}(x) = \frac{\bar{m}^2}{4} + \frac{\bar{n} - \bar{m}^2}{4}\delta_{x,0} + \frac{1-\delta_{x,0}}{4}\left[-[s_s(x)]^2 + \left(\frac{d}{dx}s_s(x)\right)\int_0^x du\, s_-(u)\right], \tag{C.1}$$

$$C^{hh}(x) = \bar{n}^2 + \bar{n}(1-\bar{n})\delta_{x,0} + (1-\delta_{x,0})\left[-[s_c(x)]^2 - \left(\frac{d}{dx}s_c(x)\right)\int_0^x du\, s_-(u)\right], \tag{C.2}$$

where $s_-(x)$ is $s_-(x) = s_s(x) - s_c(x)$. $s_\alpha(x)$ (α = c and s) are given by

$$s_\alpha(x) = \frac{\sin v_\alpha x}{\pi x}. \tag{C.3}$$

By Fourier transformation we obtain the analytic expressions of $S^{zz}(Q)$ and $N(Q)$. Taking into account of the Umklapp process, we obtain [45]

$$S^{zz}(Q) = \frac{2v_s - v_c}{4\pi} + S_I(Q) + S_I(2\pi - Q) + S_{II}(Q) + S_{II}(2\pi - Q),$$
$$N(Q) = \frac{v_c}{\pi} + N_I(Q) + N_I(2\pi - Q) + N_{II}(Q) + N_{II}(2\pi - Q),$$

for momentum $0 < Q < 2\pi$. The $S_I(Q)$ is given by

$$\begin{aligned} S_I(Q) &\equiv 2\int_0^\infty dx \cos Qx \left[-\frac{1}{4}[s_s(x)]^2 + \frac{1}{4}\left(\frac{d}{dx}s_s(x)\right)\int_0^x du\, s_s(u)\right] \\ &= \theta(2v_s - Q)\left[\frac{Q - 2v_s}{4\pi} - \frac{Q}{8\pi}\ln\left|1 - \frac{Q}{v_s}\right|\right], \end{aligned} \tag{C.4}$$

[7] In [44], the normalization factors are different. When the spin correlation is divided into $C^{zz}(x) = C^{hh}(x)/4 + C^{bh}(x) + C^{bb}(x)$, the correlation functions $C^{hh}(x)$, $C^{bh}(x)$ and $C^{bb}(x)$ correspond to $\rho_o^2 h_{oo}(x)$, $\rho_o\rho_\downarrow h_{o\downarrow}(x)$ and $\rho_\downarrow^2 h_{\downarrow\downarrow}(x)$ in [44], respectively. Tractable expressions for the correlation functions were derived by Kuramoto [45].

where $\theta(x)$ is $\theta(x) = 1$ for positive x, and 0 otherwise. This contribution has the same form as the level–level correlation of the random matrices for symplectic ensembles [23, 46, 47]. In fact, the $S^{zz}(Q)$ of the Haldane–Shastry model can be expressed by $v_s/(2\pi) + S_I(Q) + S_I(2\pi - Q)$ [48]. The $S_{II}(Q)$ contributes for finite hole doping ($\bar{n} < 1$), which is given by

$$S_{II}(Q) \equiv -\frac{1}{2}\int_0^\infty \mathrm{d}x \cos Qx \left(\frac{\mathrm{d}}{\mathrm{d}x}s_s(x)\right)\int_0^x \mathrm{d}u\, s_c(u)$$
$$= \begin{cases} \dfrac{v_c}{4\pi}, & \text{for } 0 < Q \leqslant v_s - v_c, \\ \dfrac{-Q + v_c + v_s}{8\pi} + \dfrac{Q}{8\pi}\ln\left|\dfrac{Q - v_s}{v_c}\right|, & \text{for } v_s - v_c \leqslant Q \leqslant v_s + v_c, \\ 0, & \text{for } Q \geqslant v_s + v_c. \end{cases} \tag{C.5}$$

The divergence at $Q = \pi(1 - \bar{m})$ in $S_I(Q)$ is removed by hole doping. The static spin structure factor has the same form for the Haldane–Shastry model in the region $Q \leqslant 2k_{F,\downarrow}$. This region contains '$Q \leqslant \min[\pi\bar{m}, k_{F,\downarrow}]$', where the $S^{zz}(Q,\omega)$ of the $1/r^2$ supersymmetric t–J model has the same form as that of the Haldane–Shastry model. For the momentum $0 < Q \leqslant \min[\pi\bar{m}, k_{F,\downarrow}]$, the ω-integration of the $S^{zz}(Q,\omega)$ (see equation (4)) reproduces the above expression.

Next we consider the static charge structure factor $N(Q)$. The $N_I(Q)$ is given by

$$N_I(Q) \equiv 2\int_0^\infty \mathrm{d}x \cos Qx \left[-[s_c(x)]^2 - \left(\frac{\mathrm{d}}{\mathrm{d}x}s_c(x)\right)\int_x^\infty \mathrm{d}u\, s_c(u)\right]$$
$$= \begin{cases} -\dfrac{v_c}{\pi} + \dfrac{Q}{\pi} - \dfrac{Q}{2\pi}\ln\left|1 + \dfrac{Q}{v_c}\right|, & \text{for } 0 < Q \leqslant 2v_c, \\ \dfrac{v_c}{\pi} - \dfrac{Q}{2\pi}\ln\left|\dfrac{Q + v_c}{Q - v_c}\right|, & \text{for } 2v_c \leqslant Q. \end{cases} \tag{C.6}$$

This term has the same form as the level–level correlation of the random matrices for orthogonal ensembles [46, 47]. In fact, the static structure factor of the Sutherland model with coupling parameter $\beta = 1/2$ is given by $v_c/\pi + N_I(Q)$ [23]. The $N_{II}(Q)$ is given by

$$N_{II}(Q) \equiv 2\int_0^\infty \mathrm{d}x \cos Qx \left(\frac{\mathrm{d}}{\mathrm{d}x}s_c(x)\right)\int_x^\infty \mathrm{d}u\, s_s(u)$$
$$= \begin{cases} 0, & \text{for } 0 < Q \leqslant v_s - v_c, \\ \dfrac{Q}{2\pi}\ln\dfrac{Q + v_c}{v_s} - \dfrac{Q - v_s + v_c}{2\pi}, & \text{for } v_s - v_c \leqslant Q \leqslant v_s + v_c, \\ -\dfrac{v_c}{\pi} + \dfrac{Q}{2\pi}\ln\dfrac{Q + v_c}{Q - v_c}, & \text{for } Q \geqslant v_s + v_c. \end{cases} \tag{C.7}$$

For $0 < Q \leqslant k_{F,\downarrow}$, the ω-integration of the $N(Q,\omega)$ (see equation (A.4)) reproduces the above expression.

We can rewrite $S^{zz}(Q)$ and $N(Q)$ more explicitly. Because of the reflection symmetry against $Q = \pi$, it is enough to consider $Q \leqslant \pi$. For convenience, we define the following functions:

$$S_1(x) = \frac{x}{4} - \frac{x}{8}\ln\left|\frac{1 - \bar{m} - x}{1 - \bar{m}}\right|,$$
$$S_{2a}(x) = \frac{\bar{n} - \bar{m}}{8} + \frac{x}{8} - \frac{x}{8}\ln\frac{1 - \bar{n}}{1 - \bar{m}},$$
$$S_{2b}(x) = \frac{\bar{m}}{2} - \frac{x}{8}\ln\left|\frac{\bar{m} - 1 + x}{1 + \bar{m} - x}\right| - \frac{1}{4}\ln\left|\frac{1 + \bar{m} - x}{1 - \bar{m}}\right|,$$

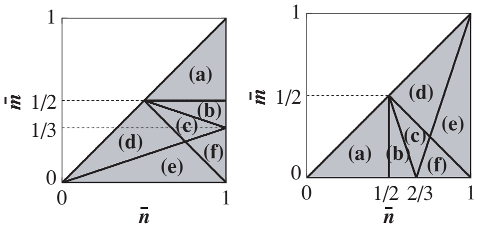

Figure C1. The shaded region shows the permissible one ($0 \leqslant \bar{n} \leqslant 1$ and $0 \leqslant \bar{m} \leqslant \bar{n}$) of the $(\bar{n}, \bar{m})$ space in the t–J model. This region can be classified into six regions with different expressions of $S^{zz}(Q)$ (left) and $N(Q)$ (right).

$$
\begin{aligned}
S_{3a}(x) &= \frac{\bar{n}-1}{4} + \frac{x}{4} - \frac{x}{8}\ln\left|\frac{1-\bar{m}-x}{1-\bar{m}}\right|, \\
S_{3b}(x) &= \frac{\bar{n}+3\bar{m}}{8} - \frac{x}{8} + \frac{x}{8}\ln\left|\frac{1+\bar{m}-x}{1-\bar{n}}\right| - \frac{1}{4}\ln\left|\frac{1+\bar{m}-x}{1-\bar{m}}\right|, \\
S_{4a}(x) &= \frac{\bar{n}-2\bar{m}+1}{4}, \\
S_{4b}(x) &= \frac{\bar{n}+2\bar{m}-1}{4} - \frac{x}{8}\ln\left|\frac{\bar{m}-1+x}{1+\bar{m}-x}\right| - \frac{1}{4}\ln\left|\frac{1+\bar{m}-x}{1-\bar{m}}\right|, \\
S_{4c}(x) &= \frac{\bar{m}}{4} - \frac{1}{4}\ln\frac{1-\bar{n}}{1-\bar{m}}.
\end{aligned}
\tag{C.8}
$$

The expressions of $S^{zz}(Q)$ can classified into six cases (a)–(f) (see figure C1 (left)).

(a) When $\bar{m} \geqslant 1/2$, we have

$$
S^{zz}(Q) = \begin{cases} S_1(Q/\pi), & \text{for } 0 < Q \leqslant \pi(\bar{n}-\bar{m}), \\ S_{2a}(Q/\pi), & \text{for } \pi(\bar{n}-\bar{m}) \leqslant Q \leqslant \pi(2-\bar{n}-\bar{m}), \\ S_{3a}(Q/\pi), & \text{for } \pi(2-\bar{n}-\bar{m}) \leqslant Q \leqslant 2\pi(1-\bar{m}), \\ S_{4a}(Q/\pi), & \text{for } 2\pi(1-\bar{m}) \leqslant Q \leqslant \pi. \end{cases}
\tag{C.9}
$$

(b) When $\bar{m} \leqslant 1/2$ and $\bar{m} \geqslant -\bar{n}/3 + 2/3$, we have

$$
S^{zz}(Q) = \begin{cases} S_1(Q/\pi), & \text{for } 0 < Q \leqslant \pi(\bar{n}-\bar{m}), \\ S_{2a}(Q/\pi), & \text{for } \pi(\bar{n}-\bar{m}) \leqslant Q \leqslant \pi(2-\bar{n}-\bar{m}), \\ S_{3a}(Q/\pi), & \text{for } \pi(2-\bar{n}-\bar{m}) \leqslant Q \leqslant 2\pi\bar{m}, \\ S_{4b}(Q/\pi), & \text{for } 2\pi\bar{m} \leqslant Q \leqslant \pi. \end{cases}
\tag{C.10}
$$

(c) When $\bar{m} \leqslant -\bar{n}/3 + 2/3$, $\bar{m} \geqslant -\bar{n} + 1$ and $\bar{m} \geqslant \bar{n}/3$, we have

$$
S^{zz}(Q) = \begin{cases} S_1(Q/\pi), & \text{for } 0 < Q \leqslant \pi(\bar{n}-\bar{m}), \\ S_{2a}(Q/\pi), & \text{for } \pi(\bar{n}-\bar{m}) \leqslant Q \leqslant 2\pi\bar{m}, \\ S_{3b}\,(Q/\pi)\,, & \text{for } 2\pi\bar{m} \leqslant Q \leqslant \pi(2-\bar{n}-\bar{m}), \\ S_{4b}(Q/\pi), & \text{for } \pi(2-\bar{n}-\bar{m}) \leqslant Q \leqslant \pi. \end{cases}
\tag{C.11}
$$

(d) When $\bar{m} \leqslant -\bar{n} + 1$ and $\bar{m} \geqslant \bar{n}/3$, we have

$$
S^{zz}(Q) = \begin{cases} S_1(Q/\pi), & \text{for } 0 < Q \leqslant \pi(\bar{n}-\bar{m}), \\ S_{2a}(Q/\pi), & \text{for } \pi(\bar{n}-\bar{m}) \leqslant Q \leqslant 2\pi\bar{m}, \\ S_{3b}\,(Q/\pi)\,, & \text{for } 2\pi\bar{m} \leqslant Q \leqslant \pi(\bar{n}+\bar{m}), \\ S_{4c}(Q/\pi), & \text{for } \pi(\bar{n}+\bar{m}) \leqslant Q \leqslant \pi. \end{cases}
\tag{C.12}
$$

(e) When $\bar{m} \leqslant -\bar{n}+1$ and $\bar{m} \leqslant \bar{n}/3$, we have

$$S^{zz}(Q) = \begin{cases} S_1(Q/\pi), & \text{for } 0 < Q \leqslant 2\pi\bar{m}, \\ S_{2b}(Q/\pi), & \text{for } 2\pi\bar{m} \leqslant Q \leqslant \pi(\bar{n}-\bar{m}), \\ S_{3b}(Q/\pi), & \text{for } \pi(\bar{n}-\bar{m}) \leqslant Q \leqslant \pi(\bar{n}+\bar{m}), \\ S_{4c}(Q/\pi), & \text{for } \pi(\bar{n}+\bar{m}) \leqslant Q \leqslant \pi. \end{cases} \tag{C.13}$$

(f) When $\bar{m} \geqslant -\bar{n}+1$ and $\bar{m} \leqslant \bar{n}/3$, we have

$$S^{zz}(Q) = \begin{cases} S_1(Q/\pi), & \text{for } 0 < Q \leqslant 2\pi\bar{m}, \\ S_{2b}(Q/\pi), & \text{for } 2\pi\bar{m} \leqslant Q \leqslant \pi(\bar{n}-\bar{m}), \\ S_{3b}(Q/\pi), & \text{for } \pi(\bar{n}-\bar{m}) \leqslant Q \leqslant \pi(2-\bar{n}-\bar{m}), \\ S_{4b}(Q/\pi), & \text{for } \pi(2-\bar{n}-\bar{m}) \leqslant Q \leqslant \pi. \end{cases} \tag{C.14}$$

To describe $N(Q)$ as well, we define the following functions:

$$\begin{aligned} N_1(x) &= x - \frac{x}{2}\ln\left|\frac{1-\bar{n}+x}{1-\bar{n}}\right|, \\ N_{2a}(x) &= \frac{\bar{n}-\bar{m}}{2} + \frac{x}{2} + \frac{x}{2}\ln\frac{1-\bar{n}}{1-\bar{m}}, \\ N_{2b}(x) &= 2 - 2\bar{n} + \frac{x}{2}\ln\left|\frac{\bar{n}-1+x}{1-\bar{n}+x}\right|, \\ N_{3a}(x) &= -\bar{m} + x - \frac{x}{2}\ln\left|\frac{1+\bar{n}-x}{1-\bar{n}}\right| + \ln\left|\frac{1+\bar{n}-x}{1-\bar{m}}\right|, \\ N_{3b}(x) &= 2 - \frac{3}{2}\bar{n} - \frac{\bar{m}}{2} - \frac{x}{2} + \frac{x}{2}\ln\left|\frac{\bar{n}-1+x}{1-\bar{m}}\right|, \\ N_{4a}(x) &= 2\bar{n} - \bar{m} + \ln\frac{1-\bar{n}}{1-\bar{m}}, \\ N_{4b}(x) &= 2 - 2\bar{n} - \bar{m} + \frac{x}{2}\ln\left|\frac{\bar{n}-1+x}{1+\bar{n}-x}\right| + \ln\left|\frac{1+\bar{n}-x}{1-\bar{m}}\right|, \\ N_{4c}(x) &= 1 - \bar{n}. \end{aligned} \tag{C.15}$$

The expressions of $N(Q)$ can be classified into six cases (a)–(f) (see figure C1 (right)).

(a) When $\bar{n} \leqslant 1/2$, we have

$$N(Q) = \begin{cases} N_1(Q/\pi), & \text{for } 0 < Q \leqslant \pi(\bar{n}-\bar{m}), \\ N_{2a}(Q/\pi), & \text{for } \pi(\bar{n}-\bar{m}) \leqslant Q \leqslant \pi(\bar{n}+\bar{m}), \\ N_{3a}(Q/\pi), & \text{for } \pi(\bar{n}+\bar{m}) \leqslant Q \leqslant 2\pi\bar{n}, \\ N_{4a}(Q/\pi), & \text{for } 2\pi\bar{n} \leqslant Q \leqslant \pi. \end{cases} \tag{C.16}$$

(b) When $\bar{n} \geqslant 1/2$ and $\bar{m} \leqslant -3\bar{n}+2$, we have

$$N(Q) = \begin{cases} N_1(Q/\pi), & \text{for } 0 < Q \leqslant \pi(\bar{n}-\bar{m}), \\ N_{2a}(Q/\pi), & \text{for } \pi(\bar{n}-\bar{m}) \leqslant Q \leqslant \pi(\bar{n}+\bar{m}), \\ N_{3a}(Q/\pi), & \text{for } \pi(\bar{n}+\bar{m}) \leqslant Q \leqslant 2\pi(1-\bar{n}), \\ N_{4b}(Q/\pi), & \text{for } 2\pi(1-\bar{n}) \leqslant Q \leqslant \pi. \end{cases} \tag{C.17}$$

(c) When $\bar{m} \geqslant 3\bar{n}-2$, $\bar{m} \leqslant -\bar{n}+1$ and $\bar{m} \geqslant -3\bar{n}+2$, we have

$$N(Q) = \begin{cases} N_1(Q/\pi), & \text{for } 0 < Q \leqslant \pi(\bar{n}-\bar{m}), \\ N_{2a}(Q/\pi), & \text{for } \pi(\bar{n}-\bar{m}) \leqslant Q \leqslant 2\pi(1-\bar{n}), \\ N_{3b}(Q/\pi), & \text{for } 2\pi(1-\bar{n}) \leqslant Q \leqslant \pi(\bar{n}+\bar{m}), \\ N_{4b}(Q/\pi), & \text{for } \pi(\bar{n}+\bar{m}) \leqslant Q \leqslant \pi. \end{cases} \tag{C.18}$$

(d) When $\bar{m} \geqslant -\bar{n} + 1$ and $\bar{m} \geqslant 3\bar{n} - 2$, we have

$$N(Q) = \begin{cases} N_1(Q/\pi), & \text{for } 0 < Q \leqslant \pi(\bar{n} - \bar{m}), \\ N_{2a}(Q/\pi), & \text{for } \pi(\bar{n} - \bar{m}) \leqslant Q \leqslant 2\pi(1 - \bar{n}), \\ N_{3b}(Q/\pi), & \text{for } 2\pi(1 - \bar{n}) \leqslant Q \leqslant \pi(2 - \bar{n} - \bar{m}), \\ N_{4c}(Q/\pi), & \text{for } \pi(2 - \bar{n} - \bar{m}) \leqslant Q \leqslant \pi. \end{cases} \tag{C.19}$$

(e) When $\bar{m} \geqslant -\bar{n} + 1$ and $\bar{m} \leqslant 3\bar{n} - 2$, we have

$$N(Q) = \begin{cases} N_1(Q/\pi), & \text{for } 0 < Q \leqslant 2\pi(1 - \bar{n}), \\ N_{2b}(Q/\pi), & \text{for } 2\pi(1 - \bar{n}) \leqslant Q \leqslant \pi(\bar{n} - \bar{m}), \\ N_{3b}(Q/\pi), & \text{for } \pi(\bar{n} - \bar{m}) \leqslant Q \leqslant \pi(2 - \bar{n} - \bar{m}), \\ N_{4c}(Q/\pi), & \text{for } \pi(2 - \bar{n} - \bar{m}) \leqslant Q \leqslant \pi. \end{cases} \tag{C.20}$$

(f) When $\bar{m} \leqslant -\bar{n} + 1$ and $\bar{m} \leqslant 3\bar{n} - 2$, we have

$$N(Q) = \begin{cases} N_1(Q/\pi), & \text{for } 0 < Q \leqslant 2\pi(1 - \bar{n}), \\ N_{2b}(Q/\pi), & \text{for } 2\pi(1 - \bar{n}) \leqslant Q \leqslant \pi(\bar{n} - \bar{m}), \\ N_{3b}(Q/\pi), & \text{for } \pi(\bar{n} - \bar{m}) \leqslant Q \leqslant \pi(\bar{n} + \bar{m}), \\ N_{4b}(Q/\pi), & \text{for } \pi(\bar{n} + \bar{m}) \leqslant Q \leqslant \pi. \end{cases} \tag{C.21}$$

In the limit $\bar{m} \to 0$, the above expressions of $S^{zz}(Q)$ and $N(Q)$ reproduce the results in [43].

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